Transfer matrix with Kogut-Susskind fermions

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A transfer matrix for gauge theories on a lattice with Kogut-Susskind fermions in the flavor basis is constructed and some difficulties in the spin-diagonal basis are pointed out.

I. INTRODUCTION

In the investigation of some features of QCD at finite baryonic density [1] we must be able to write the baryon charge operator in the transfer matrix formalism. Therefore we must express the transfer matrix in terms of quark-antiquark creation-annihilation operators. Such a formulation has been constructed by Lüscher [2] for Wilson fermions, but we have not found in the literature anything similar for Kogut-Susskind fermions. Actually we know only one paper on the subject [3], but in this work the transfer matrix is constructed in the spin-diagonal basis, where it is necessary to introduce auxiliary fields whose relation to quark-antiquarks is not transparent. Another feature, possibly related to the above, might be a difficulty in the applications we are interested in: the transfer matrix is linear in the creation-annihilation operators of quark-antiquarks, rather than exponential, so that the Hamiltonian contains the logarithm of these operators.

We have therefore tackled the problem in the flavor basis, and we have extended the formalism developed by Lüscher for Wilson fermions to the case of Kogut and Susskind.

Lüscher proved the reflection positivity [2] of the transfer matrix in the gauge $U_0=1$. This gauge fixing is too strong, but Menotti and Pelissetto [4] have extended the proof to the gauge $U_0\sim 1$, where $U_0=1$ with the exception of a single time slice.

In this paper we will ignore the pure gauge field part of the action, the partition function, and the transfer matrix. Its role in fact is the same as with Wilson fermions, and it can be included in the same way as in Lüscher's paper.

II. THE TRANSFER MATRIX

The quark field q in the flavor basis is defined on hypercubic blocks of coordinates n_μ defined in terms of the sites coordinates x_μ

$$x_\mu=2n_\mu+\eta_\mu,\quad x_\mu=0,\dots,2N_\mu-1,$$

$$n_\mu=0,\dots,N_\mu-1,\quad \eta_\mu=0,1. \quad (1)$$

It carries Dirac indices α, flavor indices f, and color indices c, sometimes comprehensively denoted in the following by I.

In the transfer matrix formalism most often one has to do with quantities at a given (Euclidean) time n_0. For this reason we use a summation convention over spatial coordinates $\mathbf{n}_j$ and intrinsic indices I at fixed time. So, for instance, we will write

$$\bar{q}(m_0)Q(m_0,n_0)q(n_0)$$

$$=\sum_{\mathbf{m},\mathbf{n},I,J}\bar{q}_{\mathbf{m},I}(m_0)Q_{\mathbf{m},I;\mathbf{n},J}(m_0,n_0)q_{\mathbf{n},J}(n_0), \quad (2)$$

where Q is the quark matrix. It is a function of time dependent link operators $U_\mu(n_0)$ which have the standard Wilson variables $U_\mu(m_0,\mathbf{m})$ as spatial matrix elements,

$$(U_\mu(m_0))_{\mathbf{m},\mathbf{n}}=\delta_{\mathbf{m},\mathbf{n}}U_\mu(m_0,\mathbf{m}). \quad (3)$$

In this notation the quark action can be written

$$S_q=\sum_{m_0,n_0}\bar{q}(m_0)Q(m_0,n_0)q(n_0). \quad (4)$$

Its explicit expression [5] is

$$S_q=\bar{q}\{M1\otimes 1+2K[\gamma_\mu\otimes 1\nabla_\mu-\gamma_5\otimes t_5t_\mu\delta_\mu]\}q, \quad (5)$$

where the lattice spacing is equal to 1, M and K are the mass and the hopping parameters, and ∇_μ and δ_μ are the derivatives

$$(\nabla_\mu f)(n)=\frac{1}{4}[f(n+2\mu)-f(n-2\mu)],$$

$$(\delta_\mu f)(n)=\frac{1}{4}[f(n+2\mu)+f(n-2\mu)-2f(n)]. \quad (6)$$

In the tensor product the γ matrices act on Dirac indices, while the t matrices act on flavor indices and are given by

$$t_\mu=\gamma_\mu^T. \quad (7)$$

The minimal gauge interaction (that where the link variables join the quark fields along the shortest path) can be introduced in the standard way by means of the block link variables U_μ.

Now we want to identify the components of the quark field which can be related to quarks and antiquarks. They propagate, respectively, forward and backward in time and

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are interchanged by charge conjugation. For this purpose we define the projection operators

$$P_\mu^{(\pm)}=\frac{1}{2}[1\otimes1\pm\gamma_\mu\gamma_5\otimes t_5t_\mu]. \tag{8}$$

We can thus rewrite the quark matrix singling out the terms that are not diagonal in time in the form

$$Q(m_0,n_0)=K[\gamma_0 P_0^{(-)}U_0(m_0)\delta_{n_0,m_0+1}$$
$$-\gamma_0 P_0^{(+)}U_0^{(+)}(n_0)\delta_{n_0,m_0-1}]+N(n_0)\delta_{m_0,n_0}, \tag{9}$$

where

$$N(n_0)=M1\otimes1+K\gamma_5\otimes t_5t_0+K\sum_{j=1}^{3}\{\gamma_5\otimes t_5t_j$$
$$+\gamma_j[P_j^{(-)}U_j(n_0)T_j^{(+)}-P_j^{(+)}T_j^{(-)}U_j^+(n_0)]\}. \tag{10}$$

The translation operators $T_j^{(\pm)}$ have matrix elements

$$(T_j^{(\pm)})_{\mathbf{m},\mathbf{n}}=\delta_{\mathbf{n},\mathbf{m}\pm e_j} \tag{11}$$

with

$$(e_j)_k=\delta_{j,k}. \tag{12}$$

We see that the components

$$q^{(\pm)}=P_0^{(\pm)}q \tag{13}$$

of the quark field propagate backward and forward respectively. It remains to check that they are properly related by charge conjugation. A possible definition of this operator, with Lüscher's convention for the γ matrices is

$$\mathcal{C}=\gamma_0\gamma_2\otimes t_0t_2 \tag{14}$$

whose action on the quark field is

$$q=\mathcal{C}^{-1}\bar{q}',$$
$$\bar{q}=-\mathcal{C}^T q'. \tag{15}$$

Therefore the charge conjugation

$$q^{(+)}=\gamma_2\otimes t_0t_2(q^{(-)})' \tag{16}$$

amounts to interchanging the $q^{(\pm)}$ with one another and to rotate the flavors. This action is acceptable insofar as the flavor is physically unobservable.

Finally, following Lüscher's procedure we define the transfer matrix $\hat{\mathcal{T}}_q$ in terms of creation and annihilation operators related to $q^{(\pm)}$ and show that

$$Z_q=Tr\hat{\mathcal{T}}_q=\int[d\bar{q}dq]\exp(-16S_q). \tag{17}$$

The factor of 16 in front of S_q is the volume of the block. We construct $\hat{\mathcal{T}}_q$ in terms of the auxiliary operator

$$\hat{T}_q(n_0)=\exp[\hat{y}N(n_0)\hat{x}] \tag{18}$$

according to the ordered product

$$\hat{\mathcal{T}}_q=\mathcal{J}\prod_{n_0=0}^{N_0-1}(\hat{T}_q(n_0))^+\hat{T}_q(n_0-1). \tag{19}$$

In the above equations $\mathcal{J}$ is the Jacobian of a transformation which will be defined later, and the operator $\hat{T}_q(n_0)$ depends on the time n_0 only through the dependence on it of the gauge fields. Because N is Hermitian, $\hat{\mathcal{T}}_q$ is positive definite. The above expression of $\hat{T}_q(n_0)$ is valid in the gauge $U_0=1$, assumed by Lüscher to prove reflection positivity. As already said, this gauge fixing is too strong, but to lighten the formalism we will nevertheless put $U_0=1$ and we will reinstate U_0 at the end. The reader can check that by keeping U_0 in the intermediate steps one arrives at the same result.

To prove Eq. (17) we must transform the trace into a Berezin integral. To this end we introduce between the factors in Eq. (19) the identity

$$1=\int[dx^+dx\,dy^+dy]\exp(-x^+x-y^+y)|x\,y\rangle\langle x\,y|, \tag{20}$$

where the basis vectors

$$|x\,y\rangle=|\exp(-x\hat{x}^+-y\hat{y}^+)\rangle \tag{21}$$

are coherent states and the x,y are Grassmann variables. We then use the following equations. First

$$\langle x|x'\rangle=\exp(x^+x'). \tag{22}$$

Second, if $B=B(\hat{x}^+)$ and $C=C(\hat{x})$ are operators that depend on $\hat{x}^+,\hat{x}$ only,

$$\langle x|B\cdot C|x'\rangle=B(x^+)\langle x|x'\rangle C(x'). \tag{23}$$

By means of these equations we can evaluate the kernel of the transfer matrix

$$\langle x(n_0),y(n_0)|\hat{T}_q(n_0)|x(n_0-1),y(n_0-1)\rangle$$
$$=\mathcal{J}^{-1}\exp[x^+(n_0)N(n_0)y^+(n_0)]$$
$$\times\exp[x^+(n_0)x(n_0-1)+y^+(n_0)y(n_0-1)]$$
$$\times\exp[y(n_0-1)N(n_0-1)x(n_0-1)]. \tag{24}$$

Collecting all the pieces we arrive at the Euclidean path integral form of the partition function

$$Z_q=\mathcal{J}^{-1}\int[dx^+dx\,dy^+dy]\exp S' \tag{25}$$

with the action

$$S' = \sum_{n_0} -x^+(n_0)x(n_0-1) - y^+(n_0)y(n_0-1)$$
$$+ x^+(n_0)x(n_0) + y^+(n_0)y(n_0) - x^+(n_0)N(n_0)y^+(n_0)$$
$$- y(n_0-1)N(n_0-1)x(n_0-1). \tag{26}$$

The change of variables

$$x = 4\sqrt{K}q^{(+)}, \quad y^+ = 4\sqrt{K}q^{(-)}, \tag{27}$$

whose Jacobian is the function $\mathcal{J}$ introduced in Eq. (19) transforms S' into S_q thus proving Eq. (17).

III. CONCLUSION

We have extended Lüscher's formalism for Wilson fermions to the case of Kogut and Susskind in the flavor basis.

This result can be sufficient for our purposes [1] but leaves unsolved the case of the spin-diagonal basis. In fact we cannot get the spin-diagonal formulation by a simple change of basis [5]. There is a practical complication in the way. To this end we should associate the gauge fields with the links of the lattice and we would get an awkward result because a minimal gauge coupling in one basis becomes cumbersome in the other [6]. One should then start from the minimal gauge coupling in the spin-diagonal basis, find the coupling resulting in the flavor basis, and show reflection positivity with such coupling. The problem requires further study and we think that deserves a wider interest than our specific motivation.

The alternative is to construct the transfer matrix directly in the spin-diagonal basis in a form convenient for the use at finite baryon density.

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